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Inverse bremsstrahlung absorption in large radiation fields during binary collisions—Born approximation I. Elastic collisions

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Abstract. The absorption of radiation by inverse bremsstrahlung absorption from a planar single mode monochromatic light beam is considered by nonrelativistic time-dependent perturbation theory, which is equivalent to the Born approximation. The results are of general validity and may be applied to any scattering system which is left unchanged by the collision. Comparison between the results obtained previously by different workers shows the relationship between the different models used to investigate this effect.

1. Introduction

For some time now there has been considerable interest in the possibility of plasma heating by lasers due to inverse bremsstrahlung. The theory of inverse bremsstrahlung absorption in a large radiation field has received some attention, and calculations have been carried out using both classical (Silin 1965, Babuel-Peyrissac 1970, Pert 1972) and quantum models (Bunkin and Federov 1966, Rand 1964). However, a detailed intercomparison is lacking. The classical theory calculations show good agreement within the limits of their approximations where they overlap. In the case of the quantum theory calculations the published results are not the same, even when taking into account the differences in the approximations made.

In this paper we develop an expression for the absorption cross section from the Born approximation, identical to that obtained by Bunkin and Federov (1966). The calculation is nonrelativistic, so that the electron velocity $v \ll c$, the velocity of light, and is performed in the dipole approximation; hence the classical electron oscillation amplitude $eE/m\omega^2 \ll c/\omega$, the wavelength of the light. The Born approximation implies the standard conditions on the electron velocity:

$$|V(r)| \ll \frac{\hbar v}{r}$$

where $V(r)$ is the scattering potential at a distance r. Using this expression the classical theory is developed as an approximation when the average electron energy gain per collision is much greater than a quantum of radiation. The results given by Bunkin and Federov have been rederived to correct a number of numerical errors and obtain better agreement with the calculations of Rand.

As an appendix the general result is also obtained using Rand's approach, which is more complicated, but closely related to classical methods (Dawson and Oberman

1962). In order to compare the results obtained by the various methods, abbreviations are used to denote the different papers, particularly with regard to equation numbers. Thus R ≡ Rand (1964), B ≡ Bunkin and Federov (1966) and P ≡ Pert (1972).

2. The Born approximation for inverse bremsstrahlung

The wavefunction for a free electron in a classical electromagnetic field of peak intensity $\boldsymbol{E}$, angular frequency ω and vector potential $\boldsymbol{A}$ can be written exactly:

$$\psi_{\boldsymbol{p}} = \frac{1}{(2\pi\hbar)^{3/2}} \exp\left[\frac{\mathrm{i}}{\hbar}\left\{\left(\boldsymbol{p}-\frac{e\boldsymbol{A}}{c}\right).\boldsymbol{r}-\int_0^t \left(\boldsymbol{p}-\frac{e\boldsymbol{A}}{c}\right)^2 \frac{\mathrm{d}t}{2m}\right\}\right]. \tag{1}$$

We wish to consider the scattering of the electron with initial momentum $\boldsymbol{p}$ into a state $\boldsymbol{p}'$ by a potential $V(r)$. Under the usual conditions for the Born approximation we consider the potential as a perturbation introducing transitions from the state $\boldsymbol{p}$ to $\boldsymbol{p}'$ by time-dependent perturbation theory. Writing the wavefunction in the potential as

$$\psi(r, t) = \int a_{\boldsymbol{p}}(t)\psi_{\boldsymbol{p}}(r, t)\, \mathrm{d}\boldsymbol{p} \tag{2}$$

and using Schrödinger's equation, we obtain

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}a_{\boldsymbol{p}'} \simeq \langle\psi_{\boldsymbol{p}'}|V|\psi_{\boldsymbol{p}}\rangle \qquad \boldsymbol{p} \neq \boldsymbol{p}' \tag{3}$$

where $\boldsymbol{p}$ is the initial state of the electron.

The transition probability into the state $\boldsymbol{p}'$ per unit time is thus

$$\frac{\mathrm{d}}{\mathrm{d}t}|a_{\boldsymbol{p}'}|^2 = a_{\boldsymbol{p}'}\frac{\mathrm{d}}{\mathrm{d}t}a^*_{\boldsymbol{p}'} + a^*_{\boldsymbol{p}'}\frac{\mathrm{d}}{\mathrm{d}t}a_{\boldsymbol{p}'}. \tag{4}$$

In order to evaluate this term we expand the wavefunction in a Fourier series in time:

$$\psi_{\boldsymbol{p}} = \psi_{\boldsymbol{p}}(r)\sum_N A_N(\boldsymbol{p}) \exp\left\{-\frac{\mathrm{i}}{\hbar}\left(\frac{p^2}{2m}+\frac{e^2E^2}{4m\omega^2}+N\hbar\omega\right)t\right\}. \tag{5}$$

Hence integrating and letting the upper limit of integration tend to infinity, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|a_{\boldsymbol{p}}|^2 = \frac{2\pi}{\hbar}|\langle\psi_{\boldsymbol{p}'}|V|\psi_{\boldsymbol{p}}\rangle|^2 \sum_n \left|\sum_N A_n(\boldsymbol{p})A^*_{n+N}(\boldsymbol{p}')\right|^2 \delta(\epsilon-\epsilon'-n\hbar\omega). \tag{6}$$

The sum over Fourier components is performed using the Faltung theorem (Morse and Feshbach 1953, p 464); and, noting that the term is real, we obtain

$$\sum_N A_N(\boldsymbol{p})A^*_{n+N}(\boldsymbol{p}') = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left\{-\frac{\mathrm{i}}{\hbar}\left(\frac{ce\boldsymbol{E}}{m\omega^2}\sin\omega t\,.(\boldsymbol{p}-\boldsymbol{p}')+n\hbar\omega t\right)\right\} \mathrm{d}(\omega t) \tag{7}$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos\left(\frac{ce\boldsymbol{E}}{m\hbar\omega^2}.(\boldsymbol{p}-\boldsymbol{p}')\sin\omega t + n\omega t\right) \mathrm{d}(\omega t). \tag{8}$$

This integral is simply the well known integral representation of the Bessel function J_n

and is evaluated in terms of polar angles defined with respect to the polar axis along $\boldsymbol{p}$ and the $(\boldsymbol{E}, \boldsymbol{p})$ azimuthal plane, and the dimensionless constants

$$\gamma = \frac{evE}{\hbar\omega^2} \qquad \xi = \frac{\hbar\omega}{mv^2} \qquad \lambda = \frac{|\boldsymbol{p}'|}{|\boldsymbol{p}|} = (1+2n\xi)^{1/2} \tag{9}$$

where v is the classical initial velocity ($= |\boldsymbol{p}|/m$), to give

$$\frac{\mathrm{d}}{\mathrm{d}t}|a_{\boldsymbol{p}'}|^2 = \frac{2\pi}{\hbar}|\langle\psi_{\boldsymbol{p}'}|V|\psi_{\boldsymbol{p}}\rangle|^2 \sum_n \mathrm{J}_n^2\gamma\{\cos\theta_0(1-\lambda\cos\theta) - \lambda\sin\theta_0\sin\theta\cos\phi\}\delta(\epsilon'-\epsilon-n\hbar\omega) \tag{10}$$

θ_0 being the polar angle of $\boldsymbol{E}$, and (θ, ϕ) the angular coordinates of $\boldsymbol{p}'$.

This expression yields the probability of a transition from the state $\psi_{\boldsymbol{p}}$ to $\psi_{\boldsymbol{p}'}$ per unit time. To obtain the electron cross section for this process, σ^{e}, we note

$$\Gamma\sigma^{\mathrm{e}} = \frac{\mathrm{d}}{\mathrm{d}t}|a_{\boldsymbol{p}'}|^2 \tag{11}$$

where Γ, the electron flux, is given by

$$\Gamma = \frac{1}{(2\pi\hbar)^3}\frac{|\boldsymbol{p}|}{m}.$$

The total cross section for all electron collisions under these conditions is thus

$$\Gamma\sigma_{\mathrm{T}}^{\mathrm{e}} = \int \frac{\mathrm{d}}{\mathrm{d}t}|a_{\boldsymbol{p}'}|^2\mathrm{d}\boldsymbol{p}' \tag{12}$$

$$\sigma_{\mathrm{T}}^{\mathrm{e}} = \sum_n \frac{2\pi}{\hbar v}\int \mathrm{d}\Omega|\langle\psi_{\boldsymbol{p}'}|V|\psi_{\boldsymbol{p}}\rangle|^2 mp'\mathrm{J}_n^2\gamma\{\cos\theta_0(1-\lambda\cos\theta) - \lambda\sin\theta_0\sin\theta\cos\phi\} \tag{13}$$

where

$$\frac{p'^2}{2m} = \frac{p^2}{2m} + n\hbar\omega.$$

This cross section represents the sum of the cross sections for the absorption of a discrete number of n photons, or emission if n is negative. The cross section for the absorption of just n photons is thus

$$\sigma_n^{\mathrm{e}} = \frac{m^2}{4\pi^2\hbar^4}\int \mathrm{d}\Omega\left|\int \mathrm{d}\boldsymbol{r}\, V(r)\exp\left(\frac{\mathrm{i}p}{\hbar}(\boldsymbol{a}-\boldsymbol{a}'\lambda)\cdot\boldsymbol{r}\right)\right|^2 \times \lambda\mathrm{J}_n^2\gamma\{\cos\theta_0(1-\lambda\cos\theta) - \lambda\sin\theta_0\sin\theta\cos\phi\} \tag{14}$$

where $\boldsymbol{a}$ and $\boldsymbol{a}'$ are unit vectors in the directions of $\boldsymbol{p}$ and $\boldsymbol{p}'$. In calculations of bremsstrahlung emission this is the quantity usually used, namely the electron collision cross section for the emission of radiation. In absorption, however, the quantity required is the absorption coefficient or photon absorption cross section σ_n, which is calculated from the number of photons absorbed per cubic centimetre, $n\rho v\sigma_n^{\mathrm{e}}$, where ρ is the electron number density. Hence

$$\sigma_n = \frac{\rho v n \sigma_n^{\mathrm{e}}}{cE^2/8\pi\hbar\omega} \tag{15}$$

$$= \frac{2m^2 v\hbar\omega\rho n}{\pi\hbar^4 E^2 c}\int \mathrm{d}\Omega \left|\int \mathrm{d}\boldsymbol{r}\, V(r)\exp\left(\frac{\mathrm{i}p}{\hbar}(\boldsymbol{a}-\boldsymbol{a}').\boldsymbol{r}\right)\right|^2$$

$$\times \lambda \mathrm{J}_n^2\gamma\{\cos\theta_0(1-\cos\theta)-\lambda\sin\theta_0\sin\theta\cos\phi\}. \tag{16}$$

It will be noted that the result quoted without derivation by Bunkin and Federov (1966, equation (B3)) differs from this by a factor of n. The factor n was introduced in equation (15) to take into account the fact that n photons are absorbed in each electron collision. The results of Bunkin and Federov (1966) and of Nicholson-Florence (1971), whose calculations are based on equation (16), thus need modifying by a factor of n.

Rand (1964) has also derived an equation analogous to (16) by a different approach. In appendix 1 equation (16) is also derived by Rand's method.

Equation (16) is of general validity provided the conditions specified earlier are obeyed. The potential $V(\boldsymbol{r})$ may be used to describe any form of scattering, namely with ions, atoms or molecules, provided the state of the scatterer is left unchanged.

3. Relation to the field-free collision cross section

3.1. *Raizer's formula*

In the absence of a radiation field we may use equation (14) to obtain the well known expression for the differential cross section in the Born approximation:

$$I(\theta,\phi) = \frac{m^2}{4\pi^2\hbar^4}\left|\int \mathrm{d}\boldsymbol{r}\, V(r)\exp\left(\frac{\mathrm{i}p}{\hbar}(\boldsymbol{a}-\boldsymbol{a}').\boldsymbol{r}\right)\right|^2. \tag{17}$$

Thus, if $\lambda \simeq 1$ and $\xi \ll 1$, then

$$\sigma_n = \frac{8\pi v\hbar\omega\rho n}{E^2 c}\int \mathrm{d}\Omega\, I(\theta,\phi)\,\mathrm{J}_n^2\gamma\{\cos\theta_0(1-\cos\theta)-\sin\theta_0\sin\theta\cos\phi\}. \tag{18}$$

In the important case of low field strengths, single photon absorption dominates. Expanding the Bessel function when $\gamma \ll 1$ we obtain

$$\sigma_1 = \frac{8\pi\hbar v\omega\rho}{E^2 c}\left(\frac{\gamma}{2}\right)^2\int \mathrm{d}\Omega\, I(\theta,\phi)\{\cos\theta_0(1-\cos\theta)-\sin\theta_0\sin\theta\cos\phi\}^2. \tag{19}$$

If $I(\theta,\phi)$ is independent of ϕ, these integrals may be simply performed to yield

$$\sigma_1 = \frac{8\pi v\hbar\omega\rho}{E^2 c}\frac{\pi}{3}\gamma^2\int I(\theta)(1-\cos\theta)\sin\theta\,\mathrm{d}\theta$$

$$= \frac{4\pi e^2 v^3\rho}{3\hbar c\omega^3}\sigma_{\mathrm{d}} \tag{20}$$

where σ_{d} is the momentum transfer cross section. This result may be immediately recognized as that obtained by Zel'dovich and Raizer (1965) using a quasi-classical semiquantitative argument which is valid for the same range of conditions as the Born approximation. Raizer's formula is thus shown to be an exact result for low fields such that single quantum absorption only is important.

Higher order terms can be calculated in a similar manner to the expansion (19). The results cannot, however, be expressed in simple terms as with equation (20).

3.2. *The classical theory*

A simple classical treatment of inverse bremsstrahlung absorption was developed in an earlier paper (Pert 1972) by considering the elastic scattering of electrons which possess both a thermal velocity $\boldsymbol{v}$ and an oscillating component $\boldsymbol{u}$ due to the field.

In this model the energy absorption ϵ per collision by an electron of constant thermal velocity varies during the cycle of the field. The probability per unit time of energy absorption in the range $\epsilon \rightarrow \epsilon + \delta\epsilon$ for electrons of constant thermal velocity scattered into a given solid angle is then

$$W_c = \frac{\omega w I(\theta', w)\, \delta\epsilon\, d\Omega'}{2\pi(d\epsilon/du)(du/dt)} \tag{21}$$

where the angle of scatter θ' and the solid angle element $d\Omega'$ are determined by the total velocity

$$\boldsymbol{w} = \boldsymbol{v} + \boldsymbol{u}$$

and

$$\epsilon = m\boldsymbol{u}(\boldsymbol{v} - \boldsymbol{v}').$$

When $\epsilon = n\hbar\omega$ and $\delta\epsilon = \hbar\omega$, we have

$$W_c \simeq \frac{wuI(\theta', w)\, d\Omega'}{\pi n u_0(1 - u^2/u_0^2)^{1/2}} \tag{22}$$

(u_0 being the oscillation velocity amplitude), provided $\epsilon \gg \hbar\omega$ (or n large)—the classical limit.

We may compare this with the value obtained from equation (14)

$$W_B = v' I(\theta', w) J_n^2\left(\frac{u_0}{u}\frac{\epsilon}{\hbar\omega}\right) d\Omega \tag{23}$$

where $d\Omega$ is defined by the thermal velocity $\boldsymbol{v}$ alone.

But

$$\frac{d\Omega}{d\Omega'} = \frac{\partial(\cos\theta, \phi)}{\partial(\cos\theta', \phi')} = \frac{w'}{v'} = \frac{w}{v'}. \tag{24}$$

Hence, using the asymptotic expansion of J_n and averaging over the rapidly varying cos term, we obtain

$$W_B \simeq \frac{wuI(\theta', w)\, d\Omega'}{\pi n u_0}. \tag{25}$$

The two terms W_B and W_c are approximately equal. The contribution from the Born approximation for $n > n_{max}$ corresponding to the maximum classical energy absorption $\epsilon_{max} = n_{max}\hbar\omega$ can be shown to be small. The argument of the Bessel function in the term, whose scattering corresponds to the classical absorption ϵ_{max}, is, from equation (23), n_{max}. Since

$$J_n(z < n) \sim 0 \tag{26}$$

terms of $n > n_{max}$ do not contribute greatly to the absorption and their effects are included in the factor $(1-u^2/u_0^2)^{1/2}$ in the denominator of equation (22). The exact equivalence of the classical and quantum mechanical models, however, cannot be directly demonstrated in this manner. A more formal proof using Rand's method, which may be derived from equation (16), is given in appendix 2.

The correspondence between the classical and quantum theories can thus be clearly seen. A collision involving the absorption of n photons is thus equivalent classically to a collision at the appropriate time during the oscillation of the electron. The equivalence of these two models allows an easy calculation of the absorption coefficient to be made in most cases of interest via the classical model. This is fortunate, as direct calculation from equation (16) is difficult due to the Bessel function term and can only be performed in the asymptotic limits or by numerical methods.

4. Electron–atom collisions

In the case of atomic collisions the matrix element can be calculated in the standard manner to yield the usual result in terms of the atomic form factor (Mott and Massey 1965, p 459). In principle the absorption coefficient could be obtained from (16); however, this is a laborious procedure and it is probably simpler to use the classical approximation.

5. Collisions with ions

Bremsstrahlung absorption in electron–ion binary collisions at high fields has already been considered by Rand (1964) and by Bunkin and Federov (1966). Numerical calculations of the Bunkin–Federov results have been performed by Nicholson-Florence (1971). However, in addition to the error already noted there are several errors in the final asymptotic results in the paper by Bunkin and Federov (1966). It is therefore of value to repeat their calculations, particularly as there is then better agreement with those of Rand (1964).

For the ions the potential $V(\boldsymbol{r}) = -Ze^2/r$ and

$$\int V(\boldsymbol{r}) \exp\left(\frac{ip}{\hbar}(\boldsymbol{a}-\boldsymbol{a}'\lambda)\,.\,\boldsymbol{r}\right) \mathrm{d}\boldsymbol{r} = \frac{2\pi\hbar^2 Ze^2}{m^2v^2\lambda\,|\cos\theta-(\lambda^2+1)/2\lambda|}. \qquad (27)\ (\mathrm{B6})$$

5.1. *Low fields*: $\gamma \ll 1$

We consider first the case of small fields, where $\gamma \ll 1$. The Bessel functions may then be replaced by their small argument expansion.

Thus for single photon absorption we have

$$\sigma_1 = \frac{8\pi Z^2 e^4 \rho \hbar\omega}{m^2 v^3 E^2 c\lambda}\left(\frac{\gamma}{2}\right)^2 I_1 \tag{28}$$

where

$$I_1 = \int_0^{2\pi}\int_{-1}^{1} \mathrm{d}\phi\,\mathrm{d}x \frac{\{\cos\theta_0(1-\lambda x)-\lambda\sin\theta_0(1-x^2)^{1/2}\cos\phi\}^2}{\{(\lambda^2+1)/2\lambda - x\}^2} \tag{29}$$

$$= 4\pi\left(\lambda^2(3\cos^2\theta_0-1)+\tfrac{1}{2}\lambda\{1+\cos^2\theta_0-\lambda^2(3\cos^2\theta_0-1)\}\ln\left|\frac{\lambda+1}{\lambda-1}\right|\right). \tag{30}$$

Consider the case of slow electrons when $\xi \simeq \lambda^2 \gg 1$, for which

$$I_1 = \frac{16\pi}{3} \tag{31}$$

and

$$\sigma_1 = \frac{2^{9/2}\pi^2 Z^2 e^6 \rho}{3m^{3/2}\hbar^{3/2}\omega^{7/2}c} \tag{32}$$

which differs from the value of Bunkin and Federov (B9) by a factor of 2 and is in exact agreement with that of Rand (R43).

For the case when $\xi \ll 1$ we obtain the same values as given by Bunkin and Federov (B10), which can be shown to be identical to those given by Rand (R31).

The second-order term in the expansion of the first-order Bessel function is of the same order as the first-order term from the second-order Bessel function in the expansion of (13). Thus the total cross section for absorption is given by

$$\sigma = \sigma_1 + \sigma_2 \tag{33}$$

$$= \frac{8\pi Z^2 e^4 \rho \hbar\omega}{m^2 c v^3 E^2}\left\{\left(\frac{\gamma}{2}\right)^2 \frac{I_1(\lambda_1)}{\lambda_1} - \left(\frac{\gamma}{2}\right)^4 \frac{I_2(\lambda_1)}{\lambda_1} + \frac{1}{2}\left(\frac{\gamma}{2}\right)^4 \frac{I_2(\lambda_2)}{\lambda_2}\right\} \tag{34}$$

and there is a similar term to include stimulated emission. I_2 is given by

$$I_2 = \int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta \frac{\{\cos\theta_0(1-\lambda\cos\theta)-\lambda\sin\theta_0\sin\theta\cos\phi\}^4}{\{(\lambda^2+1)/2\lambda-\cos\theta\}^2} \tag{35}$$

$$= 2\pi(I_2^1\cos^4\theta_0 + 3\lambda^2 I_2^2\cos^2\theta_0\sin^2\theta_0 + \tfrac{3}{8}\lambda^4 I_2^3\sin^4\theta_0) \tag{36}$$

and

$$I_2^1 = 2\lambda^6 - \frac{16}{3}\lambda^4 + 6\lambda^2 - \lambda(\lambda^2-1)^3\ln\left|\frac{\lambda+1}{\lambda-1}\right|$$

$$I_2^2 = -2\lambda^4 + \frac{10}{3}\lambda^2 + \lambda(\lambda^2-1)^2\ln\left|\frac{\lambda+1}{\lambda-1}\right| \tag{37}$$

$$I_2^3 = \frac{2\lambda^4 - 4\lambda^2/3 + 2}{\lambda^2} - \frac{(\lambda^2+1)(\lambda^2-1)^2}{\lambda^3}\ln\left|\frac{\lambda+1}{\lambda-1}\right|.$$

In the case of slow electrons we need only consider absorption, and hence

$$I_2 = \frac{16\pi\lambda^2}{5} \tag{38}$$

independent of θ_0, giving

$$\sigma = \frac{2^{9/2}\pi^2}{3} \frac{Z^2 e^6 \rho}{m^{3/2}\hbar^{3/2}\omega^{7/2}c}\left\{1 - \frac{3}{10}\left(1 - \frac{1}{\sqrt{2}}\right)\frac{e^2E^2}{m\hbar\omega^3}\right\}. \tag{39}$$

The numerical factor for the second-order term differs from that given by Bunkin and Federov (B9) by the factor of n discussed earlier, but is in good agreement with Rand, who finds a factor $\frac{9}{32}$ compared to $\frac{3}{10}$ (R43).

In many cases the electron distribution is isotropic and only the average value of I_2 over all θ_0 is required: it is easily shown that

$$\bar{I}_2 = \frac{16\pi\lambda^2}{5} \tag{40}$$

for all λ.

For comparison with Bunkin and Federov (B11), we calculate the cross section for fast electrons ($\lambda \simeq 1$) at $\theta_0 = 0$:

$$\sigma_{\pm 2} = \frac{\pi^2 Z^2 e^8 \rho v E^2}{2m^2\hbar^3\omega^7 c}\{\tfrac{8}{3} + 4(2\xi)^2 \pm \tfrac{32}{3}(2\xi)^3 \mp 8(2\xi)^3 \ln(1/\xi)\}. \tag{41}$$

For higher order effects the formula given by Bunkin and Federov (B12) needs multiplying by the factor of n discussed earlier.

5.2. *Large fields*: $\gamma \gg 1$

In the case of large fields we may expand the Bessel function by the large argument expansion, excluding the small range of solid angle where the argument is small, the value of the cross section being only weakly dependent on the cut-off provided $n\xi \gg 1$ or $u_0/v \gg 1$. In this case it is easily shown that the cross section is given by

$$\sigma_{+n}(\theta_0 = 0) = \frac{2^4\pi\omega\rho Z^2 e^3}{E^3 cn}\left(2\ln\gamma + \ln(2n\xi) + \frac{2}{n\xi}\right) \quad (42) \; (\text{B15})$$

$$\sigma_{-n}(\theta_0 = 0) = \frac{2^4\pi\omega\rho Z^2 e^3}{E^3 cn}\left\{\frac{2\lambda}{n\xi} + \ln\left(\frac{1-\lambda}{1+\lambda}\right)\right\}$$

and

$$\sigma_{\pm n}(\theta_0 = \pi/2) = \frac{2^7\pi Z^2 e^3 \rho \hbar^2 \omega^3 n}{E^3 cm^2 v^4}\frac{\lambda^2+1}{|\lambda^2 - 1|^3}\ln(\lambda\gamma). \quad (43) \; (\text{B17})$$

Thus at $\theta_0 = \pi/2$ the absorption cross section for n photons when $n\xi \ll 1$—that is, fast electrons (but with $v \ll u_0$)—is

$$\sigma_{\pm n}^{\pi/2} = \frac{2^5\pi Z^2 e^3 \rho m v^2}{E^3 c\hbar n^2}(1 \pm n\xi)(\ln\gamma + n\xi) \quad (44) \; (\text{B18})$$

and the total absorption cross section at $\theta_0 = \pi/2$ is

$$\sigma_n^{\mathrm{T}} = \frac{2^6 \pi Z^2 e^3 \rho \omega}{c E^3 n}(1 + \ln \gamma). \tag{45}$$

We may note from the discussion of the classical model that the cross section decreases rapidly when

$$n > \frac{e^2 E^2}{m \hbar \omega^3} = n_{\max} \tag{46}$$

and hence we sum σ_n^{T} to obtain the total cross section:

$$\sigma^{\pi/2} = \frac{2^6 \pi Z^2 e^3 \rho \omega}{c E^3}(1 + \ln \gamma) \ln\left(\frac{e^2 E^2}{m \hbar \omega^3}\right). \tag{47}$$

In the case of slow electrons ($\lambda \gg 1$) it is easily shown that both cross sections at 0 and $\pi/2$ reduce to

$$\sigma_n = \frac{2^5 \pi \omega \rho Z^2 e^3}{c E^3 n} \ln(\gamma \lambda) \tag{48}$$

which may be summed as for fast electrons to give

$$\sigma \simeq \frac{24 \pi \omega \rho Z^2 e^3}{E^3 c}\left\{\ln\left(\frac{e^2 E^2}{m \hbar \omega^3}\right)\right\}^2 \qquad \theta_0 = 0 \text{ or } \pi/2. \tag{49}$$

This result may be directly compared with that obtained by the classical theory, noting, however, that, due to the quantum limit at low velocities, a finite electron wavelength for small v is implied in this result. The thermal velocity cut-off must be replaced by the velocity corresponding to a wavelength equal to the oscillation amplitude and the lower impact parameter cut-off by the de Broglie wavelength corresponding to the oscillation velocity amplitude; that is, in the symbols of the previous paper,

$$v_{\mathrm{T}} = \frac{\hbar \omega^2}{e E_0} \qquad l = \frac{\hbar m \omega}{e E_0}.$$

Hence using equations (P34) and (P36) we obtain

$$\sigma = \frac{64 \pi \omega \rho Z^2 e^3}{E^3 c}\left\{\ln\left(\frac{e^2 E^2}{m \hbar \omega^3}\right)\right\}^2 \tag{50}$$

which considering the nature of the cut-offs implied above must be considered to be satisfactory agreement.

In cases where v is not small compared with u_0, the integrals are now no longer independent of the cut-offs used for the asymptotic expansion of the Bessel functions, and calculations based on this method must be considered of dubious accuracy. The classical method is therefore to be preferred in this case. In the limit of $v \gg u_0$, with $\gamma \gg 1$ (ie $m u_0^2 \gg \hbar \omega$), the integral is dominated by the region $\cos \theta \sim 1$ in the denominator of the integrand, and the small argument expansion of the Bessel function should be used, leading to equation (34). Thus, if $v \gg u_0$, equation (34) yields the cross section for the absorption independent of the value of γ. This removes an apparent contradiction between the work of Bunkin and Federov and the classical theory, and can be seen to be due to the neglect of the limit $v \ll u_0$ for equations (42)–(44) ((B15)–(B17)).

6. Conclusions

A general expression for the absorption and stimulated bremsstahlung cross sections of a nonrelativistic electron has been derived from time-dependent perturbation theory; that is, the Born approximation. The result is general and may be applied to collisions with any system which is left unchanged by the interaction. The method has been used here to consider collisions with both atoms and ions.

Inverse bremsstrahlung in large radiation fields has been previously considered by two distinct approaches: the Born approximation and the classical theory. However, comparison of the results from different papers reveals significant differences between these methods. In this paper we have shown the relation of the classical theory to the quantum calculation, and shown it to be valid in the following cases:

$$\tfrac{1}{2}mu_0^2 \gg \hbar\omega$$

and

$$\tfrac{1}{2}mu_0^2 \ll \hbar\omega \ll \tfrac{1}{2}mv^2 \tag{51}$$

the latter being Raizer's formula (Zel'dovich and Raizer 1965).

In addition, asymptotic expressions for the cross section in electron–ion collisions are derived, correcting a number of numerical errors in previous work to obtain better agreement between the calculations of Bunkin and Federov (1966) and Rand (1964), and to remove an apparent contradiction between the classical and the quantum theories in the extreme classical limit: $\tfrac{1}{2}mv^2 \gg \tfrac{1}{2}mu_0^2 \gg \hbar\omega$, suggested in the work of Bunkin and Federov (1966).

It should be noted that, despite the general validity of the results obtained here, their use is severely restricted by computational problems arising from the Bessel functions. In most practical cases of interest the classical conditions (51) are obeyed, and the classical method provides a much simpler means of calculating the absorption coefficient.

Appendix 1. Derivation of equation (16) by Rand's method

The Rand (1964) method transforms from the laboratory frame in which the ion is stationary and the electron oscillates classically to one in which the electron has only its thermal velocity and the ion oscillates. The energy transfer to the electron is then calculated from the energy lost by the ion due to the electron–ion interaction force. Or transforming back into the laboratory frame it is this energy change in the electron which is identified as the energy absorbed from the radiation field.

In the oscillating frame the electrostatic potential is given by

$$\nabla^2\phi = 4\pi e|\psi|^2 - 4\pi qS(t, \boldsymbol{r}) \tag{R1}$$

where ψ is the electron wavefunction and $qS(t, \boldsymbol{r})$ the ion charge distribution. The electron wavefunction satisfies Schrödinger's equation:

$$i\hbar\dot{\psi} = H_0\psi - e\phi'\psi. \tag{R2}$$

Let us consider the nature of the potential ϕ'. If we use the potential ϕ, given by equation (R1), there is a contribution to the total Hamiltonian from the electrostatic interaction of the electron with itself, which should not be included. To avoid all self-interaction effects ϕ' must be taken as the potential due to the ion alone. Rand, however, includes a contribution due to the perturbed electron distribution only, arguing that this introduces collective behaviour necessary for the propagation of longitudinal waves. Although on a statistical basis it may be argued that this allows the inclusion of collective effects in a cloud of electrons of constant velocity by the appropriate normalization condition, it is certainly not necessary in the consideration of binary collisions between an electron and an ion. It should be noted that, since Rand's result does not contain the plasma frequency, it does not treat collective effects properly, and also that his longitudinal waves are not necessary for the transfer of energy from the ion to the electron, as the energy transfer may take place due solely to the electrostatic field of the oscillating ion. We therefore put

$$\nabla^2\phi' = -4\pi q S(t, \boldsymbol{r}). \tag{R1'}$$

The calculation may be performed in an identical manner to Rand. Using (R4), (R8), (R15), (R16), (R17), (R18) and (R19), we obtain the energy absorbed by the system per unit time:

$$\dot{U} = -\frac{\mathrm{i}q^2\omega}{\pi^2}\sum_{n=1}^{\infty} n\int \mathrm{J}_n^2(\boldsymbol{k}\,.\,\boldsymbol{r}_0)\left(1-\frac{4\pi e^2}{k^2L^3}[\{E(p_0+\hbar\boldsymbol{k})-E(p_0)-n\hbar\omega\}^{-1}\right.$$
$$\left.+\{E(\boldsymbol{p}_0-\hbar\boldsymbol{k})-E(p_0)+n\hbar\omega\}^{-1}]\right)\frac{\mathrm{d}\boldsymbol{k}}{k^2}.$$

To evaluate this integral we note that $\dot{U}$ must be real and that $\mathrm{Im}(1/x) = \pm\pi\delta(x)$. Although the sign is not defined, from causality arguments we obtain

$$\mathrm{Im}\,\mathrm{d}(\Omega, \boldsymbol{k}) = \frac{4\pi^2e^2}{k^2L^3}[\delta\{E(\boldsymbol{p}_0+h\boldsymbol{k})-E(\boldsymbol{p}_0)-n\hbar\omega\}-\delta\{E(\boldsymbol{p}_0-\hbar\boldsymbol{k})-E(\boldsymbol{p}_0)+n\hbar\omega\}]. \tag{R23}$$

We are now in a position to compare this equation with those obtained directly from the Born approximation, namely equations (16) and (27).

Consider the first term of equation (R23) only. The delta function expresses the conservation of energy between the incoming and outgoing electron with energy gain $n\hbar\omega$. Thus

$$\hbar\boldsymbol{k} = \boldsymbol{p}'-\boldsymbol{p}$$
$$k^2 = 2|\boldsymbol{p}|^2\lambda\left(\frac{\lambda^2+1}{2\lambda}-\cos\theta\right)$$

and

$$\boldsymbol{k}\,.\,\boldsymbol{r}_0 = \frac{eE}{m\omega^2}\,.\,\frac{\boldsymbol{p}-\boldsymbol{p}'}{\hbar}$$
$$= \frac{1}{\hbar\omega}\frac{eEp}{m\omega}\{\cos\theta_0(1-\lambda\cos\theta)-\lambda\sin\theta_0\sin\theta\cos\phi\}.$$

Similarly the second term in (R23) corresponds to an energy loss of $n\hbar\omega$. Thus

$$\dot{U} = -\frac{cE^2}{8\pi}\rho L^3\sum_n(\sigma_n-\boldsymbol{\sigma}_{-n}) \tag{R28}$$

where

$$\sigma_n = \frac{8\pi q^2 e^2 \hbar\omega\rho n}{m^2 v^3} \int \mathrm{d}\Omega \frac{1}{\lambda} \frac{J_n^2 \gamma\{\cos\theta_0(1-\lambda\cos\theta) - \lambda\sin\theta_0\sin\theta\cos\phi\}}{\{(\lambda^2+1)/2\lambda - \cos\theta\}^2}$$

which is identical to the value given by equations (16) and (27) in the present paper.

Appendix 2. The classical approximation from Rand's theory

Rand has shown that for electron–ion scattering the total absorption cross section is approximately

$$\sigma = \frac{32\pi^2 e^4 q^2 \rho}{c m^3 \omega^4 r_0^2} \left\langle \frac{u^2 - \boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{u}-\boldsymbol{v}|^3} \int^{k_0} \frac{\mathrm{d}k}{k} \right\rangle. \tag{R39}$$

This result may be immediately interpreted in terms of the classical model. For electron–ion interactions the momentum-transfer cross section is given by (Sutton and Sherman 1965, p 143)

$$\sigma_{\mathrm{d}} = \frac{4\pi e^4 Z^2}{m^2 v^4} \ln\left(\frac{k_{\max}}{k_{\min}}\right).$$

Thus (R39) reduces to

$$\frac{cE^2\sigma}{8\pi} = m\rho\langle \boldsymbol{u}\cdot\boldsymbol{w} w \sigma_{\mathrm{d}}(w)\rangle \tag{P12}$$

which is the energy gain per electron per second, and which may be recognized as equation (P12) of (Pert 1972). The appropriate cut-offs have been discussed elsewhere (Rand 1964, Babuel-Peyrissac 1970, Pert 1972).

This calculation formally demonstrates the derivation of the classical formula from the Born approximation for electron–ion collisions, since equations (R17) and (16) are equivalent. The extension to arbitrary scattering potentials $V(\boldsymbol{r})$ is obvious.

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